Oxford Cambridge and RSA

Thursday 20 June 2024 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Insert

Time allowed: 2 hours

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INSTRUCTIONS

- Do **not** send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION

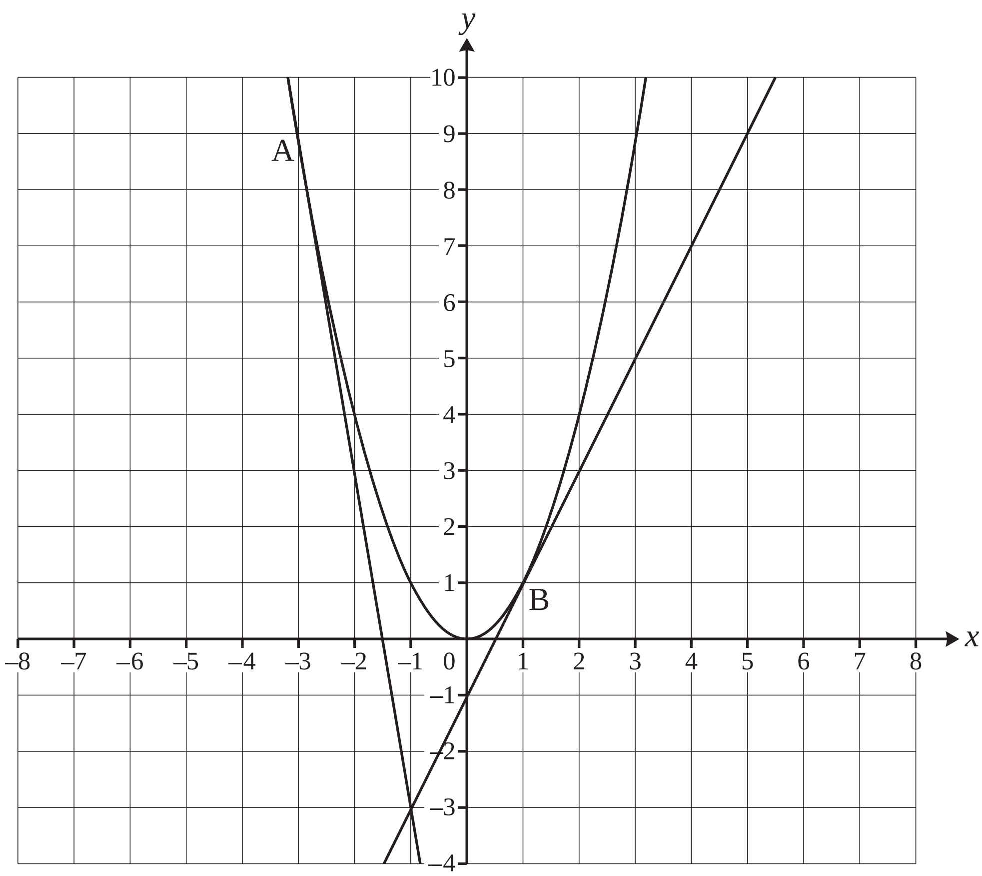
- This Insert contains the article for Section B.
- This document has **4** pages.

Tangents and normals to a quadratic curve

Tangents

Fig. C1 shows the curve $y = x^2$ together with tangents to the curve at points A $(-3, 9)$ and B $(1, 1)$. The tangents cross at the point $(-1, -3)$. This has x-coordinate -1, which is the mean of the x-coordinates of points A and B.

Fig. C1



For the curve $y = x^2$, the equation of the tangent at a general point (t, t^2) is $y = 2tx - t^2$. So the equation of the tangent at the point (t_1, t_1^2) is $y = 2t_1x - t_1^2$. There is a similar equation for the tangent at the point (t_2, t_2^2), and these two tangents cross where $2t_1x - t_1^2 = 2t_2x - t_2^2$.

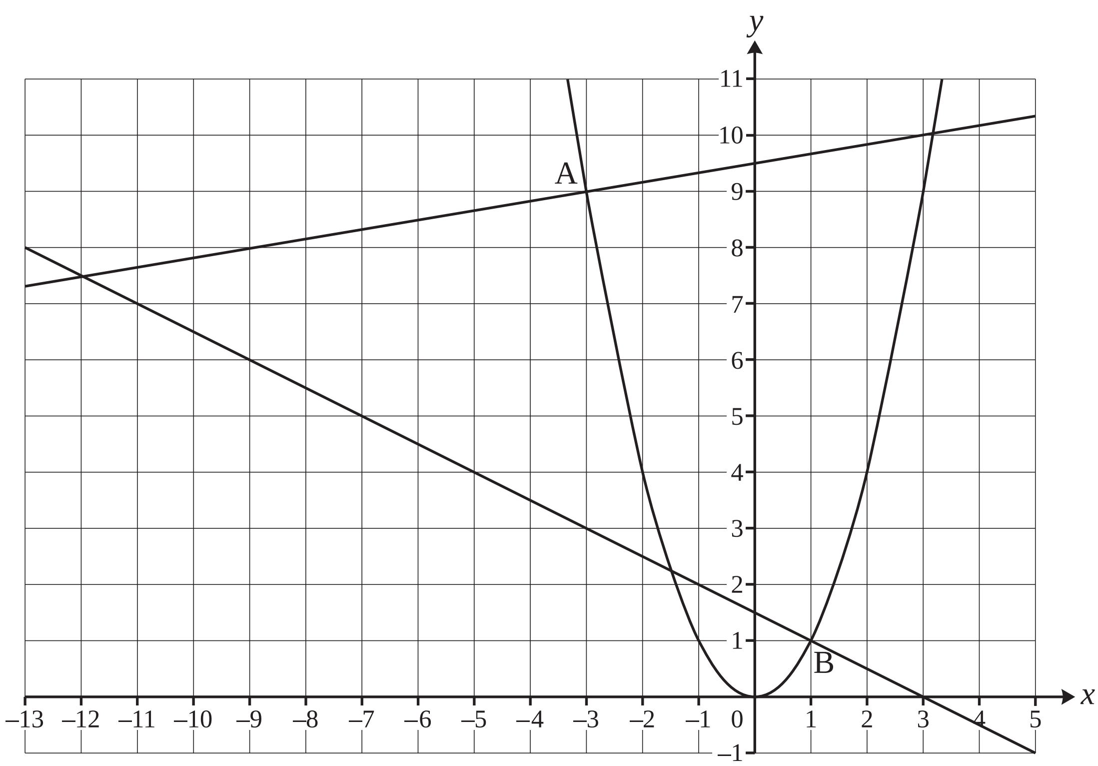
This gives $2x(t_1 - t_2) = t_1^2 - t_2^2$ so $2x(t_1 - t_2) = (t_1 - t_2)(t_1 + t_2)$ hence $x = \dfrac{t_1 + t_2}{2}$. The y-coordinate of the point of intersection is t_1t_2.

The general quadratic curve has equation $y = ax^2 + bx + c$. The tangents at any two points P and Q on this curve also cross at a point whose x-coordinate is equal to the mean of the x-coordinates of P and Q. So if P has x-coordinate x_P and Q has x-coordinate x_Q then the x-coordinate of the intersection point of the tangents is $\dfrac{x_P + x_Q}{2}$. The y-coordinate of the intersection point can be shown to be $ax_\text{P}x_\text{Q} + b\left(\frac{x_\text{P}+x_\text{Q}}{2}\right) + c$. This is equivalent to $a\left(\frac{x_\text{P}+x_\text{Q}}{2}\right)^2 + b\left(\frac{x_\text{P}+x_\text{Q}}{2}\right) + c - a\left(\frac{x_\text{P}-x_\text{Q}}{2}\right)^2$. The formula $ax_\text{P}x_\text{Q} + b\left(\frac{x_\text{P}+x_\text{Q}}{2}\right) + c$ looks simpler but $y = a\left(\frac{x_\text{P}+x_\text{Q}}{2}\right)^2 + b\left(\frac{x_\text{P}+x_\text{Q}}{2}\right) + c - a\left(\frac{x_\text{P}-x_\text{Q}}{2}\right)^2$ is in terms of the x-coordinate of the point of intersection, apart from the last term. For pairs of points with $x_\text{P} - x_\text{Q} = h$ where h is a constant, the point of intersection of the tangents lies on the curve $y = ax^2 + bx + c - \frac{ah^2}{4}$. This curve is a translation of the original curve $y = ax^2 + bx + c$.

Normals

Fig. C2 shows the curve $y = x^2$ together with normals to the curve at points A $(-3, 9)$ and B $(1, 1)$. The normals cross at the point $(-12, 7.5)$.

Fig. C2



For the curve $y = x^2$, the coordinates of the point of intersection are not as simply related to the coordinates of A and B as in the case of the tangents. The equation of the normal at the point (t, t^2) is $y = -\frac{x}{2t} + t^2 + \frac{1}{2}$. The normals at points (t_1, t_1^2) and (t_2, t_2^2) cross when $x = -2t_1t_2(t_1 + t_2)$ and $y = t_1^2 + t_2^2 + t_1t_2 + \frac{1}{2}$.